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Singularities in Complex Interfaces

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Singularities in complex interfaces

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We analyse an equation describing the motion of the material interface between two fluids in a pressure field. The interface can be expressed as the image of the unit circle under a certain time depending conformal map. This conformal transformation maps the exterior of the unit circle onto the region occupied by one of the fluids. The conformal map has singularities in the unit disc. As long as these singularities are close to the origin, the complicated non-local equation governing the evolution of the conformal map can be approximated by a somewhat simpler, local equation. We prove that there exist self-similar solutions of this equation, that they have singularities away from the origin, that these singularities hit in finite time the unit circle and that the self-similar blow up is stable to perturbations that respect the symmetry of the self-similar profile.

1. Introduction

In a recent work (Constantin & Kadanoff 1990) we derived a model equation for the interface between two fluids in a pressure field. The physical setting is that of the well known Hele-Shaw problem (Hele-Shaw 1898; Saffman & Taylor 1958; Saffman 1986; Bensimon *et al.* 1986). Two fluids, one inviscid and one viscous are confined between closely spaced glass plates. The inviscid fluid is at constant pressure. The viscous fluid is incompressible and its velocity is proportional to the gradient of its pressure. The interface moves with the fluids. The pressure jump at the interface is proportional to the curvature κ. The non-dimensionalized proportionality constant is a small parameter τ, which represents surface tension. A suction mechanism removes viscous fluid at a rate that makes the area of the region occupied by inviscid fluid grow linearly in time. Because of the incompressibility and Darcy's law, the pressure of the viscous fluid obeys Laplace's equation. This formalism leads to a eulerian description of the problem. It has the advantage of generality and the disadvantage of not describing explicitly the interface. If the suction mechanism would not be present the time derivative of the length of the boundary would be proportional to the integral of the square of the gradient of the pressure (and hence to the spatial average of the kinetic energy):

$$\frac{\mathrm{d}}{\mathrm{d}t}L = -c\int_{D_t} |\nabla p|^2 \,\mathrm{d}x, \tag{1.1}$$

where D_t is the region occupied by the viscous fluid.

This fact follows from the formula

$$\frac{\mathrm{d}}{\mathrm{d}t}A = -(\mathrm{d}-1)\int_f (v\cdot n)\, H\,\mathrm{d}S \tag{1.2}$$

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determining the evolution of the area A of a hypersurface f in $\mathbb{R}^d$ with normal n and mean curvature H when the hypersurface moves with a velocity v (P. Constantin, unpublished work). In the present case $v \cdot n = -c_1\,\partial p/\partial n$ and $H = \kappa = (\tau^{-1})\,p$, hence (1.1). The relation (1.1) is modified somewhat if a suction mechanism is present; nevertheless one may hope that it would be possible to base an existence theory of weak solutions on the relation corresponding to (1.1). A lagrangian formalism can be derived using classical potential theory: one writes p as a double layer potential; the equation $p = \tau\kappa$ on the interface becomes a Fredholm integral equation of the second kind for the potential; computing the gradient of p on the interface provides the time derivative of the interface. This lagrangian approach is well suited for the study of the well-posedness versus instability issue. One sees that, in the neighbourhood of the circular interface, the surface tension has indeed a regularizing effect, making the problem well-posed in Sobolev spaces of smooth functions, even in the unstable case. The disadvantage of the direct lagrangian approach is that the nature of the equation depends on the solution.

The conformal map formalism (Shraiman & Bensimon 1984; Tanveer 1986) describes the interface as the image of the unit circle under a time dependent conformal map. At the initial instant of time this map is assumed to have an analytic extension to the exterior of a disk of radius strictly less than one. Then, as time passes, the domain of analyticity of the conformal map may shrink. In the zero surface tension case this formalism leads to an integrable system, if the initial singularities are isolated and not essential. The singularities can move and reach the unit circle in finite time, producing a real singularity in the interface. In the case of non-zero surface tension the question of finite time singularities arising from analytic initial data is open. The equation obeyed by the conformal map is complicated: it is both highly nonlinear and highly non-local. If h denotes the derivative of the conformal map considered as a function on the unit circle in the w plane then this equation is

$$\partial h/\partial t = F(h) + \tau G(h) \tag{1.3}$$

with

$$F(h) = 2(I + \mathrm{D})(hA(1/|h|^2)) \tag{1.4}$$

and

$$G(h) = 2(I+\mathrm{D})(hA((1/|h|^2)\,\mathrm{D}(I-2A)\mathscr{K}(h))), \tag{1.5}$$

where

$$\mathscr{K}(f) = |f|^{-1}(1 + Re\,\mathrm{D}f/f), \tag{1.6}$$

$$A = \tfrac{1}{2}(I - \mathrm{i}H), \tag{1.7}$$

with H the Hilbert transform on the circle and

$$\mathrm{D}f = w\,\partial f/\partial w. \tag{1.8}$$

An exact solution of (1.3) is the circular interface

$$h = r(t), \tag{1.9}$$

where $r(t)$ is the radius of a perfectly circular bubble of area $\pi(1+2t)$. We perform a change of dependent variable

$$h = r(t)\,g \tag{1.10}$$

so that $g = 1$ is a steady solution in this new variable. Linearization about it yields

$$\partial v/\partial s = (-2I - \mathrm{D} + \tau\,\mathrm{e}^{-s}\,\mathrm{D}(\mathrm{D}^2 - I))\,v. \tag{1.11}$$

We rescaled time $s = \ln(r(t))$. In (1.11) the infinitesimal perturbation v has non-zero Fourier coefficients only for negative indices. This linear equation reveals much of the structure of the nonlinear problem. The $\tau = 0$ case is integrable but ill posed, the $\tau > 0$ case is well posed in the analytic class we are discussing, but the favourable effects of surface tension weaken as time progresses. Any initial singularity which is not placed in the origin, travels in the zero surface tension case and hits the unit circle in finite time.

In Constantin & Kadanoff (1990) we derived a local equation which approximates well the equation (1.3). We considered, in order to motivate our derivation, initial functions h, which can be extended analytically to the complement of a small disc around the origin; the complex conjugate h^* of h is finite throughout the unit circle and therefore, almost constant near the singularities of h. The approximation consists then in replacing h^* in the equation with the value it takes at the origin. The reason why the resulting equation is local (differential) is transparent once we notice that the procedure described above, when viewed as operating on functions defined on the unit circle, consists in replacing the usual product of two functions by the product of their projections on the space of functions which can be extended analytically to the exterior of the unit circle. The approximate equation is

$$\partial g/\partial s = 2-2g-\mathrm{D}g+2\tau \mathrm{D}(I-\mathrm{D}^2)\,g^{-\frac{1}{2}}. \tag{1.12}$$

It has the same linearization at $g = 1$ as (1.3), namely (1.11). The main result of Constantin & Kadanoff (1990) is theorem 1.1.

Theorem 1.1. *There exist absolute constants $\epsilon > 0$ and $C > 0$ such that, if the initial datum*

$$g_0(w) = 1+\sum_{j=0}^{\infty} g_j(0)\,w^{-j}$$

satisfies

$$\sum_{j=0}^{\infty} |g_j(0)|\rho^j \leqslant \epsilon$$

for some $\rho > 0$, then, for any $\tau \geqslant 0$ there exists a unique global solution of (1.12)

$$g^{(\tau)}(w,s) = 1+\sum_{j=0}^{\infty} g_j(s)\,w^{-j}$$

defined for all $s \geqslant 0$, $\rho|w| > \mathrm{e}^s$. This solution satisfies

$$\sum_{j=0}^{\infty} |g_j(s)|(\rho\,\mathrm{e}^{-s})^j \leqslant C\epsilon.$$

The solution corresponding to $\tau = 0$ is explicit,

$$g^{(0)}(w,s) = 1+\mathrm{e}^{-2s}\sum_{j=0}^{\infty} g_j(0)\,\mathrm{e}^{js}\,w^{-j}$$

and the difference $g^{(\tau)}-g^{(0)}$ satisfies

$$\sup_{\{\rho'|w|\geqslant \mathrm{e}^s\}} |g^{(\tau)}(w,s)-g^{(0)}(w,s)| \leqslant \epsilon \min\{\mathrm{e}^{-s}\,\tau C(1-(\rho'/\rho))^{-4}; 3\}$$

for any $\rho' \leqslant \rho$.

This result shows that no singularity in the solution of the $\tau > 0$ problem arises before the moment a singularity forms in the corresponding solution of the $\tau = 0$ problem. At this moment the difference between solutions is small away from the singularity.

2. Existence of self-similar solutions

We recall the equation derived in Constantin & Kadanoff (1990)

$$r^2\,\partial g/\partial t = 2-2g-\mathrm{D}g+2\tau r^{-1}\,\mathrm{D}(I-\mathrm{D}^2)\,g^{-\frac{1}{2}} \tag{2.1}$$

with the condition

$$g(\infty) = 1. \tag{2.2}$$

The quantity r is the radius of a circle of area $\pi(2t+1)$. The operator D is

$$\mathrm{D} = w\,\partial/\partial w \tag{2.3}$$

and the solutions g we seek are analytic in the exterior of a closed neighbourhood of the origin contained in the unit circle. We change the time variable to

$$s = \mathrm{e}^r \tag{2.4}$$

and replace g by

$$u = 1-g. \tag{2.5}$$

The equation for u is

$$\partial u/\partial s = -2u-\mathrm{D}u+2\tau e^{-s}\,\mathrm{D}(I-\mathrm{D}^2)(1+u)^{-\frac{1}{2}} \tag{2.6}$$

with the condition $u(\infty) = 0$. We drop e^{-s} in the above equation and obtain a more dissipative autonomous equation:

$$\partial u/\partial s = -2u-\mathrm{D}u+2\tau\mathrm{D}(I-\mathrm{D}^2)(1+u)^{-\frac{1}{2}}. \tag{2.7}$$

In Constantin & Kadanoff (1990) we proved that (2.1) has solutions for initial data close to 1 and these solutions remain close to the corresponding solution of the $\tau = 0$ problem up to the time when the latter becomes singular. Now we are investigating the nature of the singularities in the $\tau > 0$ case. We study (2.7) and make the self-similar ansatz:

$$u(w,s) = H(\mathrm{e}^{\omega s}\,w^{-k}). \tag{2.8}$$

Because we are interested in the region $|w| \geqslant 1$ we will seek functions $H(z)$ which are analytic in a neighbourhood of the origin (which contains $|z| \leqslant 1$). Because we need single valued functions of w in a neighbourhood of ∞ we assume that $k \geqslant 1$ is an integer. The time growth rate ω is a real number. The condition at infinity for u becomes

$$H(0) = 0. \tag{2.9}$$

The equation for H is

$$(2I+(\omega-k)\,\mathrm{D})\,H = 2\tau k\mathrm{D}(k^2\mathrm{D}^2-I)(1+H)^{-\frac{1}{2}}. \tag{2.10}$$

The function $H \mapsto (1+H)^{-\frac{1}{2}}$ is analytic for H small and

$$(1+H)^{-\frac{1}{2}} = 1-\tfrac{1}{2}H+f(H) \tag{2.11}$$

with

$$f(H) = \sum_{j=2}^{\infty} c_j\,H^j, \tag{2.12}$$

$$c_j = (-1)^j\,(2j)\,!/(2^j j\,!)^2, \tag{2.13}$$

$$\lim_{j\to\infty} |c_j|\,j^{\frac{1}{2}} = \pi^{\frac{1}{2}}. \tag{2.14}$$

The solution H of (2.10) solves

$$[2I+(\omega-k(1+\tau))\,\mathrm{D}+\tau k^3\mathrm{D}^3]\,H = 2\tau k\mathrm{D}(k^2\mathrm{D}^2-I)\,f(H) \tag{2.15}$$

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in a neighbourhood of the origin. The equation (2.15) has only the trivial analytic solution $H = 0$ unless the linear operator

$$L = 2I + (\omega - k(1+\tau))\,\mathrm{D} + \tau k^3 \mathrm{D}^3 \tag{2.16}$$

annihilates the function z. (This is a simple exercise using the method of Constantin & Kadanoff (1990).) The requirement above is equivalent to the dispersion relation

$$\omega = -2 + k(1+\tau) - \tau k^3. \tag{2.17}$$

Let us consider the symbol of the operator L, call it $m(j)$. It is defined for positive integers j and, if we require (2.17) – as we must – then $m(j)$ is given by

$$m(j) = 2 - (2 + \tau k^3)\,j + \tau k^3 j^3, \tag{2.18}$$

or, better,

$$m(j) = \tau k^3 (j-1)(j^2 + j - (2/\tau k^3)). \tag{2.19}$$

Now, clearly,

$$\text{if } \tau k^3 \geqslant 1, \quad \text{then} \quad m(j) \geqslant \tau k^3 j^3 \quad \text{for} \quad j \geqslant 2. \tag{2.20}$$

On the other hand, for τk^3 small, the equation $m(j) = 0$ has solutions if

$$\tau k^3 \in \{(2/j(j+1))\,|\,j \in \mathbb{N},\, j \geqslant 2\}.$$

We consider the set

$$A = (0,1) \setminus \{2/j(j+1)\,|\,j \in \mathbb{N},\, j \geqslant 2\}. \tag{2.21}$$

For any $x \in A$ there exists ϵ positive, depending on x, such that

$$|x - 2/j(j+1)| \geqslant \epsilon/j \quad \text{for all} \quad j \geqslant 2. \tag{2.22}$$

Consequently

$$\text{if} \quad \tau k^3 \in A, \quad \text{then} \quad |m(j)| \geqslant \epsilon(j^2 - 1) \quad \text{for} \quad j \in \mathbb{N}, j \geqslant 2. \tag{2.23}$$

If either $\tau k^3 \geqslant 1$ or $\tau k^3 \in A$ then we will seek solutions H of (2.15) (and hence of (2.10)) of the form

$$H(z) = cz + V_c(z), \tag{2.24}$$

where $c \in \mathbb{C}$ is a small parameter and the analytic function V_c satisfies

$$V_c(0) = V_c'(0) = 0. \tag{2.25}$$

The equation for V_c is

$$V = 2(I+K)\,f(cz+V), \tag{2.26}$$

where the linear operator K has symbol $a(j)$ given by

$$a(j) = ((2 + \tau k(k^2 - 1))\,j - 2)/m(j) \tag{2.27}$$

defined for $j \geqslant 2$, $j \in \mathbb{N}$. We consider, for fixed ρ the Banach algebra B_ρ used in Constantin & Kadanoff (1990) and take its closed subalgebra C_ρ of functions satisfying (2.25). We recall that this simply means that we are considering functions

$$V(z) = \sum_{j=2}^{\infty} v_j z^j \tag{2.28}$$

satisfying

$$\|V\|_\rho = \sum_{j=2}^{\infty} |v_j| \rho^j < \infty. \tag{2.29}$$

The operator K is compact in all the algebras C_ρ. The nonlinear functional

$$V \mapsto \mathscr{F}(c, V) = V - 2(I+K)\,f(cz+V) \tag{2.30}$$

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is defined for any $c \in \mathbb{C}$, $|c| \leqslant d$ and $V \in C_\rho$, $\|V\|\rho \leqslant d$ for small $d > 0$. Moreover, it is analytic in both variables and satisfies

$$\mathscr{F}(0,0) = 0 \tag{2.31}$$

and

$$\delta\mathscr{F}/\delta V(0,0) = I. \tag{2.32}$$

By the implicit function theorem there exists a unique complex curve V_c of solutions of

$$\mathscr{F}(c, V) = 0, \tag{2.33}$$

defined for c in a neighbourhood of 0 in $\mathbb{C}$. The solutions V_c can also be obtained for fixed c, $|c| \leqslant d$, via the rapidly converging iteration

$$V^{(n+1)} = 2(I+K)f(cz+V^{(n)}), \tag{2.34}$$

with $V^{(0)}$ small enough because, for small d, the map

$$V \mapsto 2(I+K)f(cz+V) \tag{2.35}$$

is a contraction in the ball of radius d in C_ρ.

Thus, if $\tau k^3 \in A$ or if $\tau k^3 \geqslant 1$ then (2.15) has a unique solution of the form (2.24) which is analytic in a neighbourhood of 0. (Unique because the coefficients in the series expansion are uniquely determined by c.) On the other hand, if $\tau k^3 = 2/l(l+1)$ for some $l \geqslant 2$ then we cannot find solutions of the form (2.24). If we look instead for solutions of the form $H = cz^l + V$ with V analytic and satisfying $V(0) = V'(0) = \ldots V^{(l)}(0) = 0$ we can repeat the whole procedure and find a complex curve of solutions. It turns out, however, that these are not new solutions. Indeed, one can easily see that V must be of the form $V(z) = W(z^l)$ and thus $H(z) = G(z^l)$ where G is the solution of (2.15) corresponding to k replaced by kl. (Note that the time growth $\omega(kl)$ corresponding to kl equals in this case l times the time growth corresponding to k.) Therefore the solution $u(w,s) = H(\mathrm{e}^{\omega s}\, w^{-k})$ corresponding to k is the solution of the same form corresponding to kl. Note that $\tau(kl)^3 \geqslant 1$.

For fixed τ and k the curve of solutions which we found is actually determined from one such solution by dilations. The equation (2.10) is dilation invariant and therefore if H is a solution so is $H_c(z) = H(cz)$ for any complex constant c.

Theorem 2.1. *The equation* (2.7) *has solutions of the form*

$$u(w,s) = H(c\,\mathrm{e}^{\omega s}\, w^{-k})$$

for any complex constant c, any values of the surface tension τ and any positive integer k. The function H is analytic in a neighbourhood of the origin. The time growth rate is given by

$$\omega = -2 + k(1+\tau) - \tau k^3.$$

If τk^3 is not of the form $2/l(l+1)$ for l an integer, $l \geqslant 2$ then H is uniquely determined by the conditions

$$H(0) = 0, \quad H'(0) = 1.$$

If $\tau k^3 = 2/l(l+1)$ the solution is the same as the one corresponding to the integer kl.

3. Formation of singularities

We will describe conditions which guarantee that the functions H determined by the equations

$$[2I - (2+\tau k(k^2-1))\,\mathrm{D}]H = 2\tau k\mathrm{D}(k^2\mathrm{D}^2 - I)(1+H)^{-\frac{1}{2}}, \tag{3.1}$$

$$H(0) = 0, \quad H'(0) = 1, \tag{3.2}$$

have a finite radius of convergence. In the previous section we saw that H exists and is analytic in a small neighbourhood of the origin if τk^3 is not of the form $2/l(l+1)$ for some integer $l \geqslant 2$. In particular, H admits a power series expansion near the origin

$$H(z) = z + az^2 + \dots, \tag{3.3}$$

where the constant a is given by

$$a = 3\tau k(4k^2-1)/4(3\tau k^3-1). \tag{3.4}$$

Note that $a > 0$ if, and only if, the positive integer k satisfies

$$\tau k^3 > \tfrac{1}{3}. \tag{3.5}$$

This is a significant condition because $\frac{1}{3} = \max\{2/l(l+1) | l \geqslant 2\}$. Another meaning of (3.5) is the following: (3.5) holds if and only if the symbol $m(j)$ of the operator L (equations (2.16)–(2.18)) is positive for all $j \geqslant 2$. It will turn out that this is also a condition of stability of H.

Theorem 3.1. *If condition* (3.5) *above is satisfied and* $k \geqslant 2$, *then the solution* H *of* (3.1), (3.2) *is not entire.*

Theorem 3.2. *If the inequalities* (3.5), $k \geqslant 2$, *and*

$$\omega = -2 + k(1+\tau) - \tau k^3 > 0$$

are satisfied then there exist self-similar solutions

$$u(w, s) = H_c(e^{\omega s} w^{-k})$$

of (2.7) *which, at time* $s = 0$ *are analytic in an open domain containing the exterior of the unit circle in the* w *plane and which cease to be so at a finite time* $s_* > 0$.

Theorem 3.2 is a straightforward consequence of theorem 3.1. The rest of this section is devoted to the proof of theorem 3.1.

The reasoning will be by contradiction. Denote

$$F = (1+H)^{-\frac{1}{2}} \tag{3.6}$$

so that

$$H = F^{-2} - 1. \tag{3.7}$$

First we note that the solution H is real for real z. We will prove that the function $F(x)$ must vanish for some positive x. Where F vanishes, H is infinite (equation (3.7)). The equation (3.1) is

$$2D^3F = \frac{2}{k^2}DF - \left(\frac{2}{\tau k^3} + \frac{k^2-1}{k^2}\right)DH + \frac{2}{\tau k^3}H. \tag{3.8}$$

We will use the notation D^{-1} for the operation defined on functions V which vanish at the origin by the conditions

$$D(D^{-1}V) = V \tag{3.9}$$

and

$$D^{-1}V(0) = 0. \tag{3.10}$$

Thus, if

$$V(z) = \sum_{j=1}^{\infty} v_j z^j \tag{3.11}$$

then

$$D^{-1}V(z) = \sum_{j=1}^{\infty} j^{-1} v_j z^j. \tag{3.12}$$

Also, for real $x > 0$, D^{-1} is given by

$$\mathrm{D}^{-1}V(x) = \int_0^x y^{-1}V(y)\,dy, \tag{3.13}$$

and consequently $\mathrm{D}^{-1}V \geqslant 0$ if $V \geqslant 0$ on an interval on the real positive semiaxis.

Applying D^{-1} to both sides of (3.8) we get

$$2\mathrm{D}^2F = \frac{2}{k^2}(F-1) - \left(\frac{2}{\tau k^3} + \frac{k^2-1}{k^2}\right)H + \frac{2}{\tau k^3}\mathrm{D}^{-1}H. \tag{3.14}$$

We will multiply (3.14) by $\mathrm{D}F$ and apply again D^{-1}. To prepare for that, we note that

$$(\mathrm{D}F)\,\mathrm{D}^{-1}H = \mathrm{D}(F\mathrm{D}^{-1}H) - FH \tag{3.15}$$

and that

$$H\mathrm{D}F = -\mathrm{D}(F + F^{-1} - 2). \tag{3.16}$$

We introduce the function

$$G = F + F^{-1} - 2 + \mathrm{D}^{-1}((\mathrm{D}F)(\mathrm{D}^{-1}H)). \tag{3.17}$$

We obtain from (3.14)

$$(\mathrm{D}F)^2 = (F-1)^2(k^{-2} + (1-k^{-2})\,F^{-1}) + 2G/\tau k^3. \tag{3.18}$$

Now we investigate the function G. First we note that

$$\mathrm{D}G = -\mathrm{D}F(H - \mathrm{D}^{-1}H). \tag{3.19}$$

Because $H(x) = x + ax^2 + \dots$, it follows that $F(x) = 1 - \frac{1}{2}x + \dots$, and consequently $\mathrm{D}F(x) = -\frac{1}{2}x + \dots$. Therefore, for small $x > 0$, $\mathrm{D}F(x) < 0$. On the other hand $H(x) - \mathrm{D}^{-1}H(x) = \frac{1}{2}ax^2 + \dots$ and, because of our assumption that $a > 0$ it follows that $H - \mathrm{D}^{-1}H > 0$ for small $x > 0$. From (3.19) we deduce that $\mathrm{D}G$ is positive for small $x > 0$ and therefore so is G. From (3.18) it follows that the set

$$X = \{x > 0 \mid \mathscr{E}(y) > 0, \quad \text{for all} \quad 0 < y \leqslant x\} \tag{3.20}$$

is not empty. We used the notation

$$\mathscr{E} = -\mathrm{D}F - (1-F)(k^{-2} + (1-k^{-2})\,F^{-1})^{\frac{1}{2}}. \tag{3.21}$$

We wish to argue that X must be the whole real positive semiaxis, unless F^{-1} becomes infinite at finite x. To do this we write $\mathrm{D}G$ in the form

$$\mathrm{D}G = 2(-\mathrm{D}F)\,\mathrm{D}^{-1}(F^{-3}(-\mathrm{D}F - \tfrac{1}{2}F(1-F^2))) \tag{3.22}$$

and then in the form

$$\mathrm{D}G = 2(-\mathrm{D}F)\,\mathrm{D}^{-1}(F^{-3}(\mathscr{E} + \mathscr{F})) \tag{3.23}$$

with

$$\mathscr{F} = (1-F)\frac{k^{-2} + (1-k^{-2})\,F^{-1} - \frac{1}{4}(F(1+F))^2}{(k^{-2} + (1-k^{-2})\,F^{-1})^{\frac{1}{2}} + \frac{1}{2}F(F+1)}. \tag{3.24}$$

Consider $\alpha = \sup X$. If $0 < x < \alpha$ then F is decreasing, so the denominator of $\mathscr{F}$ is increasing. But the value of this denominator at $x = 0$ is zero, so it follows that

$$\mathscr{F}(x) \geqslant 0 \quad \text{for} \quad 0 < x < \alpha. \tag{3.25}$$

From (3.23) it follows that

$$\mathrm{D}G(x) \geqslant 0 \quad \text{for} \quad 0 < x < \alpha. \tag{3.26}$$

Then, in view of the fact that $DG(x) > 0$ for small x it follows that, if α is finite, then

$$G(\alpha) > 0. \tag{3.27}$$

But, because of (3.18) this would imply that

$$\mathscr{E}(\alpha) > 0 \tag{3.28}$$

which would permit the extension of the inequality $\mathscr{E} > 0$ beyond α, contradicting the way α was defined. Thus $\alpha = \infty$ and so

$$-DF > (1-F)(k^{-2}+(1-k^{-2})\,F^{-1})^{\frac{1}{2}} \tag{3.29}$$

holds for all $0 < x$. But this is again absurd because any positive solution F of the inequality (3.29) which starts out the way F does, must vanish at a finite $x_* > 0$. Indeed, denoting

$$M(x) = (k^{-2}+(1-k^{-2})\;F^{-1}(x))^{\frac{1}{2}}, \tag{3.30}$$

a direct computation shows that (3.29) is equivalent to

$$2(1-k^{-2})\frac{DM}{(M^2-1)(M^2-k^{-2})} > 1. \tag{3.31}$$

This can be integrated

$$\frac{d}{dx}\ln\left(\frac{M-1}{M+1}\left(\frac{M-k^{-1}}{M+k^{-1}}\right)^{-k}\right) > \frac{d}{dx}\ln x. \tag{3.32}$$

In view of the behaviour of M at 0,

$$M(0) = 1\;(dM/dx)\,(0) = \tfrac{1}{4}(1-k^{-2}), \tag{3.33}$$

it follows from (3.32) that the function

$$x^{-1}\frac{M-1}{M+1}\left(\frac{M-k^{-1}}{M+k^{-1}}\right)^{-k} \tag{3.34}$$

is increasing for positive x and has a finite limit C_k at $x = 0$. Therefore

$$\frac{M-1}{M+1}\left(\frac{M-k^{-1}}{M+k^{-1}}\right)^{-k} > C_k\,x \tag{3.35}$$

for $0 < x$, where

$$C_k = \tfrac{1}{8}(1-k^{-2})\left(\frac{k+1}{k-1}\right)^k. \tag{3.36}$$

Now, (3.35) is absurd because its left-hand side is an increasing function of M on the interval $1 \leqslant M \leqslant \infty$ and takes the value 1 at $M = \infty$. Thus, the left-hand side of (3.35) is bounded above by 1 while the right-hand side is unbounded. This concludes the proof of theorem 3.1.

As a by-product of the proof we obtain an upper bound on the radius of convergence of the solution. Indeed, this radius cannot exceed the value of x which turns (3.35) into an equality for $M = \infty$. This value is

$$x_* = C_k^{-1}. \tag{3.37}$$

The limit of C_k as $k \to \infty$ is $\tfrac{1}{8}e^2$, which implies that the radii of convergence of all the solutions H under consideration are uniformly bounded above, for all admissible τ, k.

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4. Stability of the self-similar blow up

Consider the equation (2.7) and one of the self-similar solutions (2.8). If we perform the change

$$u(w, s) = h(e^{\omega s} w^{-k}, s) \tag{4.1}$$

then the difference

$$v(y, s) = h(y, s) - H(y) \tag{4.2}$$

obeys the equation

$$\partial v/\partial s = -L(v - 2(I+K)(f(h) - f(H))), \tag{4.3}$$

where f, L, K are given in (2.12), (2.16), (2.27). We write

$$f(h) - f(H) = \phi v \tag{4.4}$$

with the function ϕ given by

$$\phi = \int_0^1 f'((1-p)H + ph)\,dp \tag{4.5}$$

(the integral is in B_ρ). Setting

$$b = -2(I+K)(\phi v) \tag{4.6}$$

the equation (4.3) is equivalent to the integral equations

$$v_j(s) = e^{-m(j)s} v_j(0) + \int_0^s e^{-m(j)(s-\sigma)} b_j(\sigma)\,d\sigma, \tag{4.7}$$

relating the coefficients v_j and b_j of the power series expansions of v and b. Because the operator $(I+K)$ is bounded and diagonal (acts as multiplication by $(1+a(j))$ we obtain

$$\sup_{0\leqslant s\leqslant S}|v_j(s)| \leqslant |v_j(0)| + C\sum_{l=1}^{j-1}(\sup_{0\leqslant s\leqslant S}|\phi_l(s)|)(\sup_{0\leqslant s\leqslant S}|v_{j-l}(s)|) \tag{4.8}$$

for all $j \geqslant 2$, provided we require (3.5) so that $m(j) > 0$. Actually, the inequality is valid at $j = 1$ as well because the integral term in (4.7) is absent in that case. Multiplying by ρ^j and summing we obtain

$$\|v\|_{\rho, S} \leqslant \|v_0\|_\rho + C\|\phi\|_{\rho, S}\|v\|_{\rho, S} \tag{4.9}$$

for all $S > 0$, where the norms are given by

$$\|v\|_{\rho, S} = \sum_{j=1}^{\infty}(\sup_{0\leqslant s\leqslant S}|v_j(s)|)\,\rho^j. \tag{4.10}$$

The constant C does not depend on either S or ρ. The spaces $B_{\rho, S}$ defined by these norms are Banach algebras and were used for the proof of theorem 1.1 in Constantin & Kadanoff (1990). As long as $\|v\|_{\rho, S} + \|H\|_\rho$ is less than a small number δ (for instance half the radius of convergence of (2.12)) it follows that

$$\|\phi\|_{\rho, S} \leqslant \|f'\|_\delta. \tag{4.11}$$

Because $f'(0) = 0$, by taking δ small enough we ensure $2C\|f'\|_\delta \leqslant 1$ and then we deduce from (4.9) that

$$\|v\|_{\rho, S} \leqslant 2\|v_0\|_\rho. \tag{4.12}$$

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Now we choose ρ and $\|v_0\|_\rho$ small enough so that

$$\|H\|_\rho + 2\|v_0\|_\rho \leqslant \delta. \tag{4.13}$$

It follows that (4.12) holds for all $S > 0$. We proved the following theorem.

Theorem 4.1. *Assume that the numbers* τ, k *and* $\omega = -2 + k(1+\tau) - \tau k^3$ *satisfy the inequalities*

$$\tau k^3 > \tfrac{1}{3}, \quad k \geqslant 2, \quad \omega > 0.$$

Then there exists $\rho > 0$ *such that the solution* H *of* (3.1), (3.2) *is a nonlinearly stable steady solution of the equation*

$$\partial h/\partial s = -(2I + (\omega - k)\,\mathrm{D})\,h + 2\tau k \mathrm{D}(k^2\,\mathrm{D}^2 - \mathrm{I})(1+h)^{-\frac{1}{2}}$$

in the norm $\|\cdot\|_\rho$.

If the initial datum for the equation (2.7) is close at time $s = 0$ to the self-similar profile $H_c(w^{-k})$ and is a function of w^{-k}, then the solution stays close to the self-similar solution $H_c(\mathrm{e}^{\omega s} w^{-k})$ for all s in a region in the w plane, which recedes in time.

5. Conclusions

We found solutions of the local equation (2.7) that develop singularities in finite time for $\tau > 0$ and arbitrarily small or large. These solutions are self-similar and, in a certain sense, the blow up is stable: when viewed in the similarity variables, the self-similar solutions are nonlinearly stable. In contrast with the zero surface tension case where any local singularity can occur, in the positive τ case the isolated and non-essential singularities seem to be $\frac{4}{3}$ branch cuts. This number comes out of a simple minded and yet inescapable balance of singular terms argument.

The local equation derived by us in Constantin & Kadanoff (1990) fails to capture the non-local interactions of singularities. If the solutions to the full problem do not develop singularities for positive τ these non-local interactions must be responsible for the regularity.

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